Name Solutions

Class	Register Number

4049/02 **22/S4PR/AM/2**

ADDITIONAL MATHEMATICS **PAPER 2**

Tuesday **30 August 2022** **2 hours 15 minutes**

VICTORIA SCHOOL

PRELIMINARY EXAMINATION
SECONDARY FOUR

Candidates answer on the Question Paper.

No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your name, class and register number in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

Setters: Ms Emmeline Lau and Mdm Ernie Bte Abdullah

This paper consists of **18** printed pages, including the cover page.

[Turn over

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Theorem

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$
$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) = \cos A\cos B \quad \sin A\sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \quad \tan A\tan B}$$
$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\Delta = \frac{1}{2}ab\sin C$$

1 Show that $x-1$ is a factor of $2x^3-x^2-3x+2$ and hence solve the equation $2x^3-x^2-3x+2=0$ completely. [5]

Let $\text{f}(x)$ be $2x^3-x^2-3x+2$.

$\text{f}(1)=2(1)^3-(1)^2-3(1)+2=0$

By factor theorem, $x-1$ is a factor of $\text{f}(x)$.

$2x^3-x^2-3x+2=(x-1)(2x^2+bx-2)$

Comparing coefficient of $x^2: b-2=-1$

$b=1$

$2x^3-x^2-3x+2=0$

$(x-1)(2x^2+x-2)=0$

$x=1$ or $2x^2+x-2=0$

$$x=\frac{-1\pm\sqrt{1^2-4(2)(-2)}}{2(2)}$$

$$=\frac{-1\pm\sqrt{17}}{4}$$

$\therefore x=1$ or $x\approx-1.28$ or $x\approx0.781$

2 (a) Show that the equation $2\text{e}^x-1=3\text{e}^{-x}$ has only one solution and find its exact value. [4]

$2\text{e}^x-1=3\text{e}^{-x}$

$2\text{e}^x-1=\frac{3}{\text{e}^x}$

Let $u=\text{e}^x$

$2u-1=\frac{3}{u}$

$2u^2-u-3=0$

$(2u-3)(u+1)=0$

$u=\frac{3}{2}$ or $u=-1$

$\text{e}^x=\frac{3}{2}$ or $\text{e}^x=-1$ (rejected as $\text{e}^x>0$ for all real x)

$\therefore 2\text{e}^x-1=3\text{e}^{-x}$ has only one solution (shown)

$\text{e}^x=\frac{3}{2}$

$x=\ln\frac{3}{2}$

(b) Explain how the solution of $2\text{e}^{\ln 2x}-1=3\text{e}^{\ln\frac{1}{2x}}$ can be deduced from your answer in part **(a)** and find the solution. [2]

$2\text{e}^{\ln 2x}-1=3\text{e}^{\ln\frac{1}{2x}}$

$2\text{e}^{\ln 2x}-1=3\text{e}^{-\ln 2x}$

The solution of $2\text{e}^{\ln 2x}-1=3\text{e}^{\ln\frac{1}{2x}}$ can be found by replacing x in part **(a)** with $\ln 2x$.

$\ln 2x=\ln\frac{3}{2}$

$2x=\frac{3}{2}$

$x=\frac{3}{4}$

3 **(a)** Given that $y = x\sqrt{4x-3}$, show that $\frac{dy}{dx} = \frac{6x-3}{\sqrt{4x-3}}$. [3]

$$\frac{dy}{dx} = \sqrt{4x-3} + x \cdot \frac{1}{2}(4x-3)^{-\frac{1}{2}}(4)$$
$$= \frac{(4x-3)+2x}{\sqrt{4x-3}}$$
$$= \frac{6x-3}{\sqrt{4x-3}} \text{ (shown)}$$

(b) Hence find the value of $\int_1^7 \frac{6x}{\sqrt{4x-3}}\,dx$. [5]

From part **(a)** $\frac{d}{dx}\left(x\sqrt{4x-3}\right) = \frac{6x-3}{\sqrt{4x-3}}$

$$\int_1^7 \frac{6x-3}{\sqrt{4x-3}}\,dx = \left[x\sqrt{4x-3}\right]_1^7$$
$$\int_1^7 \frac{6x}{\sqrt{4x-3}}\,dx = \left[x\sqrt{4x-3}\right]_1^7 + \int_1^7 3(4x-3)^{-\frac{1}{2}}\,dx$$
$$= \left[x\sqrt{4x-3}\right]_1^7 + \left[\frac{3(4x-3)^{\frac{1}{2}}}{\frac{1}{2}(4)}\right]_1^7$$
$$= \left[x\sqrt{4x-3}\right]_1^7 + \left[\frac{3(4x-3)^{\frac{1}{2}}}{2}\right]_1^7$$
$$= \left[7\sqrt{4(7)-3} - (1)\sqrt{4(1)-3}\right] + \frac{3}{2}\left[\sqrt{4(7)-3} - \sqrt{4(1)-3}\right]$$
$$= 40$$

4 An ant moves in a straight line such that, t seconds after leaving a fixed point O, its velocity is modelled by $v = 8 + 2t - t^2$.

(a) Find the velocity of the ant when its acceleration is 1 cm/s^2. [3]

Let the acceleration of the ant be a.

$v = 8 + 2t - t^2$

$a = 2 - 2t$

$2 - 2t = 1$

$t = \frac{1}{2}$ s

$v = 8 + 2\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = 8.75$ cm/s

(b) Find the distance travelled by the ant in the first 5 seconds. [5]

$v = 8 + 2t - t^2 = 0$

$t^2 - 2t - 8 = 0$

$(t+2)(t-4) = 0$

$t = -2$ (reject) or $t = 4$

Distance travelled in the first 5 s

$$= \int_0^4 8 + 2t - t^2\,dt - \int_4^5 8 + 2t - t^2\,dt$$
$$= \left[8t + t^2 - \frac{t^3}{3}\right]_0^4 - \left[8t + t^2 - \frac{t^3}{3}\right]_4^5$$
$$= \left[\left(8(4) + 4^2 - \frac{4^3}{3}\right) - 0\right] - \left[\left(8(5) + 5^2 - \frac{5^3}{3}\right) - \left(8(4) + 4^2 - \frac{4^3}{3}\right)\right]$$
$= 30$ cm

Alternative for distance travelled

$s = \int 8 + 2t - t^2\,dt$

$= 8t + t^2 - \frac{t^3}{3} + c$

When $t = 0$, $s = 0 \Rightarrow c = 0$

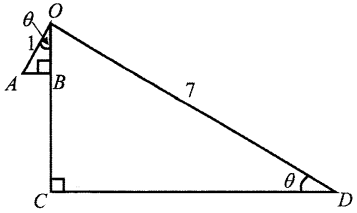
$s = 8t + t^2 - \frac{t^3}{3}$

When $t = 4$, $s = 8(4) + (4)^2 - \frac{(4)^3}{3} = 26\frac{2}{3}$

When $t = 5$, $s = 8(5) + (5)^2 - \frac{(5)^3}{3} = 23\frac{1}{3}$

Total distance travelled $= 26\frac{2}{3} + \left(26\frac{2}{3} - 23\frac{1}{3}\right) = 30$ cm

5



The diagram above shows the plan of a yard.
It is given that angle ODC = angle $AOB = \theta$, $OD = 7$ m and $OA = 1$ m.
AB and CD are each perpendicular to OC. A fence is to be built along AB, BC and CD.

(i) Show that $AB + BC + CD = (8\sin\theta + 6\cos\theta)$ m. [3]

$$\sin\theta = \frac{AB}{1} = AB$$
$$\cos\theta = \frac{OB}{1} = OB$$
$$\sin\theta = \frac{OC}{7}$$
$$OC = 7\sin\theta$$
$$\cos\theta = \frac{CD}{7}$$
$$CD = 7\cos\theta$$
$$\begin{aligned} AB + BC + CD &= AB + (OC - OB) + CD \\ &= \sin\theta + (7\sin\theta - \cos\theta) + 7\cos\theta \\ &= (8\sin\theta + 6\cos\theta) \text{ m (shown)} \end{aligned}$$

(ii) Express $AB + BC + CD$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0° \le \alpha \le 90°$. [2]

$$\begin{aligned} AB + BC + CD &= 8\sin\theta + 6\cos\theta \\ &= R\sin(\theta + \alpha) \end{aligned}$$
$$R = \sqrt{8^2 + 6^2} = 10$$
$$\tan\alpha = \frac{6}{8}$$
$$\begin{aligned} \alpha &= \tan^{-1}\left(\frac{3}{4}\right) \\ &\approx 36.870° \\ &\approx 36.9° \end{aligned}$$
$$\therefore AB + BC + CD \approx 10\sin(\theta + 36.9°)$$

(iii) Explain why the length of the fence needed can never be 11 m. [1]

Since the maximum value of $\sin(\theta + 36.9°)$ is 1,
the maximum value of $AB + BC + CD \approx 10\sin(\theta + 36.9°) = 10\,(1) = 10$.
Therefore $AB + BC + CD$ can never be 11 m.

(iv) Find the values of θ for which the length of the fence is 8.5 m. [3]

$$10\sin(\theta + 36.870°) = 8.5$$
$$\sin(\theta + 36.870°) = 0.85$$
$$\begin{aligned} \text{Basic angle} &= \sin^{-1}(0.85) \\ &\approx 58.212° \end{aligned}$$
Since $\sin(\theta + 36.870°)$ is positive, θ is acute and $36.870° \le \theta + 36.870° \le 126.870°$,
$$\theta + 36.870° \approx 58.212°, 180° - 58.212°$$
$$\theta \approx 21.3°, 84.9°$$

6 **(a)** Show that the second term in the expansion, in ascending powers of x, of $\left(2+\frac{x}{8}\right)^n$, is $n2^{n-4}x$, where n is a positive integer greater than 2 and find the third term in a similar form. [4]

$$\left(2+\frac{x}{8}\right)^n = 2^n + \binom{n}{1}2^{n-1}\left(\frac{x}{8}\right) + \binom{n}{2}2^{n-2}\left(\frac{x}{8}\right)^2 + \ldots$$

$$= 2^n + \binom{n}{1}2^{n-1}\left(\frac{x}{2^3}\right) + \binom{n}{2}2^{n-2}\left(\frac{x^2}{2^6}\right) + \ldots$$

$$= 2^n + n2^{n-1-3}x + \frac{n(n-1)}{2}2^{n-2-6}x^2 + \ldots$$

$$= 2^n + n2^{n-4}x + n(n-1)2^{n-9}x^2 + \ldots$$

Second term $= n2^{n-4}x$ (shown)

Third term $= n(n-1)2^{n-9}x^2$

(b) The first two terms in the expansion, in ascending powers of x, of $(1-x)\left(2+\frac{x}{8}\right)^n$ are $p+qx^2$, where p and q are constants.

(i) Show that the value of n is 16. [3]

$$(1-x)\left(2+\frac{x}{8}\right)^n$$

$$= (1-x)\left[2^n + n2^{n-4}x + \ldots\right]$$

$$= 2^n + n2^{n-4}x - x2^n + \ldots$$

$$= 2^n + \left(n2^{n-4} - 2^n\right)x + \ldots$$

Comparing coefficient of x:

$$n2^{n-4} - 2^n = 0$$

$$n2^{n-4} = 2^n$$

$$n = \frac{2^n}{2^{n-4}}$$

$$= 16 \text{ (shown)}$$

(ii) Hence find the value of p and of q. [2]

$$p = 2^{16} = 65\,536$$

$$(1-x)\left(2+\frac{x}{8}\right)^{16}$$

$$= (1-x)\left[2^{16} + (16)2^{12}x + 16(15)2^7x^2 + \ldots\right]$$

Comparing coefficient of x^2:

$$q = 16(15)2^7 - (16)2^{12}$$

$$= -34816$$

7 **(a)** The population of cheetahs, P, in n years, can be modelled by $P = ab^n$, where a and b are constants. Explain how a straight line graph can be drawn to represent the formula, and state how the values of a and b could be obtained from the line. [3]

$P = ab^n$

$\ln P = \ln(ab^n)$

$= \ln a + \ln b^n$

$\ln P = (\ln b)n + \ln a$

When we plot **ln *P* against *n***,
a straight line graph can be drawn to represent the formula.

$\ln b =$ gradient of the line

$b = e^{\text{gradient of the line}}$

$\ln a = \ln\text{-}P$ intercept

$a = e^{\ln\text{-}P \text{ intercept}}$

Alternative Plot lg *P* against *n*

(b) Drone A moves along a horizontal straight line. Its displacement, s m, from a fixed point O, t seconds after it passes through O is recorded in the table below.

s	10	32	66	112
t	2	4	6	8

A physicist believed that these figures can be modelled by $s = ut + \frac{1}{2}at^2$, where u is the initial velocity of Drone A and a is its constant acceleration.

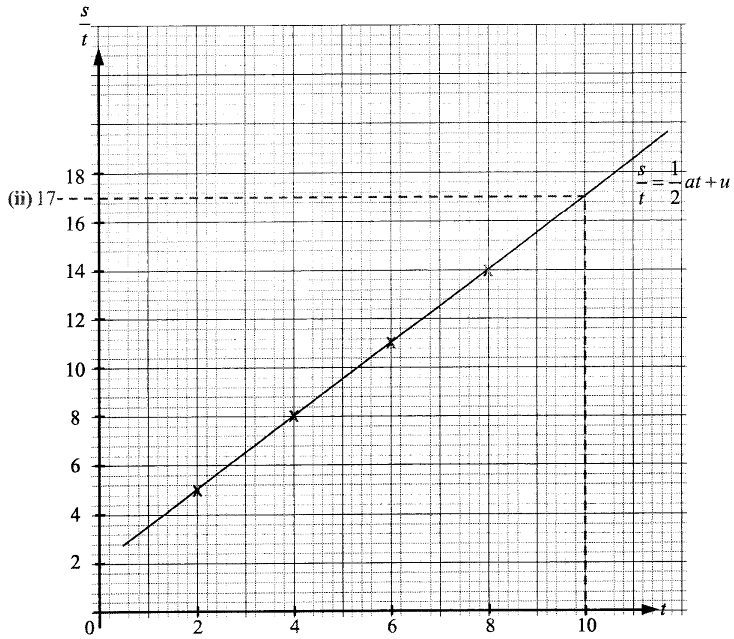
$s = ut + \frac{1}{2}at^2$

$\frac{s}{t} = \frac{1}{2}at + u$

Plot $\frac{s}{t}$ against t.

$\frac{s}{t}$	5	8	11	14
t	2	4	6	8

(i) Draw a straight line graph to show that the model is reasonable. [4]



(ii) Use your graph to estimate the displacement of Drone A when $t = 10$. [1]

When $t = 10$, $\frac{s}{t} = 17$

$s = 170$ m

(iii) Drone B moves along the same horizontal straight line as Drone A from O four seconds after Drone A. Its displacement, s m, from O, t seconds after Drone A passes through O can be modelled by $s = 3t^2 - 12t$. By using your graph in part **(i)**, explain how you can estimate when the drones will meet. [2]

$s = 3t^2 - 12t$

$\frac{s}{t} = 3t - 12$

Add the line $\frac{s}{t} = 3t - 12$ onto the graph in part **(i)**

The ***t*-coordinate** of the point of intersection of the two lines will be when the drones will meet.

8 **(a)** Show that the equation

$$(p+1)x^2+(p+3)x-(p+2)=0$$

has two real roots for all real values of p. [4]

$(p+1)x^2+(p+3)x-(p+2)=0$

$(p+1)x^2+(p+3)x+[-(p+2)]=0$

Discriminant:

$(p+3)^2-4(p+1)[-(p+2)]$

$=(p+3)^2+4(p+1)(p+2)$

$=p^2+6p+9+4p^2+12p+8$

$=5p^2+18p+17$

$=5\left(p^2+\frac{18}{5}p\right)+17$

$=5\left(p+\frac{9}{5}\right)^2-5\left(\frac{9}{5}\right)^2+17$

$=5\left(p+\frac{9}{5}\right)^2+\frac{4}{5}$

Since $\left(p+\frac{9}{5}\right)^2\geq 0$ for all real values of p,

Discriminant $=5\left(p+\frac{9}{5}\right)^2+\frac{4}{5}>0$ for all real values of p.

Thus the equation has two real roots for **all real values of p**.

(b) The equation of a curve is $y=3x^2-5ax+2a^2$, where a is a positive constant.

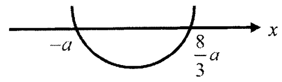
(i) Find, in terms of a, the set of values of x for which the curve lies below the line $y=10a^2$ and represent this set on a number line. [4]

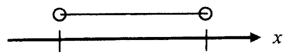
$3x^2-5ax+2a^2<10a^2$

$3x^2-5ax-8a^2<0$

$(3x-8a)(x+a)<0$

$-a<x<\frac{8}{3}a$





(ii) Find the value of a for which the curve touches the line $y=1-3ax$. [3]

$3x^2-5ax+2a^2=1-3ax$

$3x^2-2ax+2a^2-1=0$

Discriminant = 0

$(-2a)^2-4(3)(2a^2-1)=0$

$4a^2-24a^2+12=0$

$a^2=\frac{3}{5}$

$a=\sqrt{\frac{3}{5}}$

$=0.775$ (to 3sf)

9 The equation of a circle C, with centre O, is $x^2+y^2-4x-6y-5=0$.

(i) Find the coordinates of O and the exact radius of C. [3]

$x^2+y^2-4x-6y-5=0$

$2g=-4 \qquad 2f=-6 \qquad c=-5$

$g=-2 \qquad f=-3$

Coordinates of $O=(2,3)$

Radius of $C=\sqrt{(-2)^2+(-3)^2-(-5)}=\sqrt{18}=3\sqrt{2}$ units

OR

$x^2+y^2-4x-6y-5=0$

$\left(x^2-4x\right)+\left(y^2-6y\right)-5=0$

$\left[(x-2)^2-2^2\right]+\left[(y-3)^2-3^2\right]-5=0$

$(x-2)^2+(y-3)^2=18$

Coordinates of $O=(2,3)$

Radius of $C=\sqrt{18}=3\sqrt{2}$ units

The line l is a tangent to the circle at the point $P(5,6)$.

(ii) Find the equation of l. [3]

Gradient of $OP=\frac{6-3}{5-2}=\frac{3}{3}$

Gradient of $l=-1$

Equation of l:

$y-6=-(x-5)$

$y=-x+11$

(iii) Points A and B are on C such that AB is a diameter of C and is also parallel to l. Find the equation of AB. [2]

Equation of AB:

$y-3=-(x-2)$

$y=-x+5$

(iv) Hence find the coordinates of A and of B. [4]

$x^2+y^2-4x-6y-5=0$ ---(1)

$y=-x+5$ --- (2)

Substitute (2) into (1):

$x^2+(5-x)^2-4x-6(5-x)-5=0$

$x^2+25-10x+x^2-4x-30+6x-5=0$

$2x^2-8x-10=0$

$x^2-4x-5=0$

$(x+1)(x-5)=0$

$x=-1$ or $x=5$

From (2):

$y=6 \qquad y=0$

The coordinates of A and B are $(-1,6)$ and $(5,0)$.

Alternative

$y=-x+5$ ---(1)

$\sqrt{(x-2)^2+(y-3)^2}=\sqrt{18}$ ---(2)

Substitute (1) into (2),

$\sqrt{(x-2)^2+(-x+5-3)^2}=\sqrt{18}$

$(x-2)^2+(2-x)^2=18$

$2(x-2)^2=18$

$(x-2)^2=9$

$x-2=\pm 3$

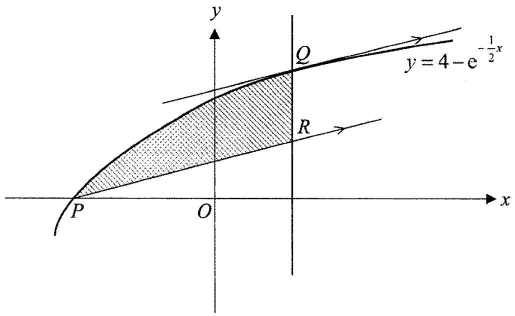
$x=-1$ or $x=5$

$y=-(-1)+5 \qquad y=-(5)+5$

$=6 \qquad =0$

The coordinates of A and B are $(-1,6)$ and $(5,0)$.

10



The diagram shows part of a curve with equation $y=4-e^{-\frac{1}{2}x}$ meeting the x-axis at the point P. A line $x=2\ln 2$ intersects the curve at the point Q. R is a point on the line $x=2\ln 2$ such that PR is parallel to the tangent to the curve at Q. Show that the area of the shaded region is $a(\ln 2)^2+b\ln 2-c$, where a, b and c are constants to be determined. [12]

$y=4-e^{-\frac{1}{2}x}$

At P, $4-e^{-\frac{1}{2}x}=0$

$e^{-\frac{1}{2}x}=4$

$-\frac{1}{2}x=\ln 4$

$x=-2\ln 4=-4\ln 2$

$\frac{dy}{dx}=\frac{1}{2}e^{-\frac{1}{2}x}$

When $x=2\ln 2$, $\frac{dy}{dx}=\frac{1}{2}e^{-\frac{1}{2}(2\ln 2)}=\frac{1}{4}$

Equation of PR:

$y-0=\frac{1}{4}(x+2\ln 4)$

$y=\frac{1}{4}x+\frac{1}{2}\ln 4$ or $y=\frac{1}{4}x+\ln 2$

When $x=2\ln 2$, $y=\frac{1}{4}(2\ln 2)+\ln 2=\frac{3}{2}\ln 2$

Alternative to find the y-coordinate of R Let $R(2\ln 2, y_R)$.

$\frac{y_R-0}{2\ln 2-(-4\ln 2)}=\frac{1}{4}$

$y_R=\frac{6\ln 2}{4}$

$=\frac{3}{2}\ln 2$

Continuation of working space for Question 10.

Area of shaded region

$=\int_{-4\ln 2}^{2\ln 2} 4-e^{-\frac{1}{2}x}\,dx-\frac{1}{2}(2\ln 2+4\ln 2)\left(\frac{3}{2}\ln 2\right)$

$=\left[4x-\frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}}\right]_{-4\ln 2}^{2\ln 2}-\frac{9}{2}(\ln 2)^2$

$=\left[4x+2e^{-\frac{1}{2}x}\right]_{-4\ln 2}^{2\ln 2}-\frac{9}{2}(\ln 2)^2$

$=\left(4(2\ln 2)+2e^{-\frac{1}{2}(2\ln 2)}\right)-\left(4(-4\ln 2)+2e^{-\frac{1}{2}(-4\ln 2)}\right)-\frac{9}{2}(\ln 2)^2$

$=\left(-\frac{9}{2}(\ln 2)^2+24\ln 2-7\right)$ units^2

Alternative

Area of shaded region

$=\int_{-4\ln 2}^{2\ln 2}\left(4-e^{-\frac{1}{2}x}\right)-\left(\frac{1}{4}x+\ln 2\right)dx$

$=\left[4x-\frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}}\right]_{-4\ln 2}^{2\ln 2}-\left[\frac{1}{8}x^2+(\ln 2)x\right]_{-4\ln 2}^{2\ln 2}$

$=\left[4x+2e^{-\frac{1}{2}x}\right]_{-4\ln 2}^{2\ln 2}-\left[\left(\frac{1}{8}(2\ln 2)^2+(\ln 2)(2\ln 2)\right)-\left(\frac{1}{8}(-4\ln 2)^2+(\ln 2)(-4\ln 2)\right)\right]$

$=\left(4(2\ln 2)+2e^{-\frac{1}{2}(2\ln 2)}\right)-\left(4(-4\ln 2)+2e^{-\frac{1}{2}(-4\ln 2)}\right)-\frac{9}{2}(\ln 2)^2$

$=\left(-\frac{9}{2}(\ln 2)^2+24\ln 2-7\right)$ units^2

End of Paper